# Math 1320: Solving Exponential and Logarithmic Equations

**What is a logarithmic function?** Previously, we have worked with exponential functions, functions that contain a variable in the exponent. Consider the following exponential equations:

$$4^x = 16 \qquad\qquad 2^x = 8 \qquad\qquad 4^x = 26$$

If we were asked to find the value of $x$ for each of the equations above, the first and second are straightforward, since $16 = 4^2$ and $8 = 2^3$. What about equation three? What power of 4 equals 26? This equation isn't so simple.

Exponential functions are one-to-one, meaning each input ($x$) has a unique output ($y$), and therefore have an inverse function, called logarithmic functions. Because of this relationship, we can convert between the exponential and logarithmic forms:

| **Logarithmic Functions** | **Exponential Functions** |
|---|---|
| $log_b x = y$ | $b^y = x$ |
| "The log, base $b$, of $x$ is $y$" | "$b$ to the power of $y$ is $x$" |

Notice that the answer to a logarithmic equation gives us an exponent. Now we can solve $4^x = 26$ by converting the equation to a log form.

⋆ Note: There are two special logarithms:

$$\log_{10} = \log \qquad \text{[common logarithm]}$$
$$\log_e = \ln \qquad \text{[natural logarithm]}$$

**Properties of Logarithms**

| **Common Logarithms** | **Natural Logarithms** |
|---|---|
| $log\ 1 = 0$ | $ln\ 1 = 0$ |
| $log\ 10 = 1$ | $ln\ e = 1$ |
| $log\ 10^x = x$ | $ln\ e^x = x$ |
| $10^{log\ x} = x$ | $e^{ln\ x} = x$ |

**Using Logarithms to Solve Exponential Equations**

1. Isolate the exponential expression.
2. Take the common logarithm on both sides of the equation for base 10. Take the natural logarithm on both sides of the equation for bases other than 10.
   [e.g. $10^x = 3 \to \log 10^x = \log 3 \to x = \log 3$
   $2^x = 7 \to \ln 2^x = \ln 7 \to x \ln 2 = \ln 7$]
3. Simplify using one of the following properties:
   $\ln b^x = x \ln b$ or $\ln e^x = x$ or $\log 10^x = x$
4. Solve for the variable.

**When do we use this?**

If we have an exponential equation, we want to use logarithms to get the the variable out of the exponent. Then we can solve the equation as we have in the past.

**Example 1. Solving Exponential Equations**
Solve: $2e^{3x-1} - 5 = 11$

**Using the Definition of a Logarithm to Solve Logarithmic Equations**

1. Express the equation in the form $\log_b M = c$.
2. Use the definition of a logarithm to rewrite the equation in exponential form:
   $\log_b M = c \qquad \text{means} \qquad b^c = M$
3. Solve for the variable.
4. Check proposed solutions in the original equation. Include in the solution set only values for which $M > 0$.

**When do we use this?**
If we have a logarithmic equation, we need to rewrite in exponential form, then we can solve the equation as we have in the past.

**Example 2. Solving Logarithmic Equations**
Solve: $\log_2(x - 1) = 4$

**Practice Problems**

Solve each equation. If an exponential equation, express the solution set in terms of natural logarithms or common logarithms. Use a calculator to find the decimal approximation for the solution, rounding to two decimal places.

1. $3^x = 21$ $[x = \frac{\ln 21}{\ln 3} \approx 2.77]$
2. $5e^{2x} = 20$ $[x = \frac{\ln 4}{2} \approx 0.69]$
3. $log_5(3x - 4) = 3$ $[x = 43]$
4. $2\ln(4x) = 10$ $[x = \frac{e^5}{4} \approx 37.10]$